Hence $R_{xx}(\tau,\zeta)$ is completely determined if it is known on a a strip $0 \leqq \tau < \infty, t \leqq \zeta < t + 2T$.

In the special case of weakly periodic nonstationary white noise excitation, i.e.,

$$R_{ff}(t_1, t_2) = F(t_1)\delta(t_1 - t_2) \tag{21}$$

where $F(t_1) = F(t_1 + T)$ and $\delta(\ )$ is the Dirac delta function, Eq. (17) gives

$$Z^{-1}(t+T,t)R_{xx}(t,t) - R_{xx}(t,t)Z^T(t+T,t) = \int_t^{t+T} Z(t,s)F(s)Z^T(t+T,s)ds \tag{22}$$

and Eq. (19) reduces to

$$R_{xx}(t_1,t_2) = Z(t_1,t_2)R_{xx}(t_2,t_2), t_1 > t_2 \tag{23}$$

For the white noise case, a differential equation satisfied by $R_{xx}(t,t)$ is given in Refs. 3 and 4. Equation (22) gives the periodic solution of that equation. Also, the preceding analysis shows that the direct time domain approach is not limited to white noise excitation.

Spectral Density

$R_{ff}(\tau,\zeta)$ is, for fixed τ, periodic in ζ with period $2T$. Hence, if $R_{ff}(\tau,\zeta)$ is absolutely integrable in τ, then it is apparent that $R_{ff}(\tau,\zeta)$ has the spectral representation

$$R_{ff}(\tau,\zeta) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{ff}(\omega,n)e^{i\omega\tau}e^{in\pi\zeta/T}d\omega \tag{24}$$

Inversion yields

$$\Phi_{ff}(\omega,n) = \frac{1}{4\pi T}\int_{-\infty}^{\infty}\int_{-T}^{T} R_{ff}(\tau,\zeta)e^{-i\omega\tau}e^{-in\pi\zeta/T}d\zeta d\tau \tag{25}$$

Substituting Eq. (24) into Eq. (7), using Eq. (4) and rearranging terms gives

$$R_{xx}(t_1,t_2) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} H(t_1,\frac{n\pi}{T} - \omega)\Phi_{ff}(\omega,n)H^T(t_2,\frac{n\pi}{T} + \omega)\,d\omega \tag{26}$$

where

$$H(t,\lambda) = \int_{-\infty}^{t} Z(t,s)e^{i\lambda s}ds \tag{27}$$

Equation (26) is a generalization of the mixed time frequency relation used in Ref. 1 to include weakly periodic nonstationary excitation.

If $R_{ff}(t_1,t_2)$ can be expressed in the product form

$$R_{ff}(t_1,t_2) = F(t_1)R(t_1 - t_2)G^T(t_2) \tag{28}$$

where $R(t_1 - t_2)$ has a Fourier transform $\Phi(\omega)$, then

$$R_{xx}(t_1,t_2) = \int_{-\infty}^{\infty} H_1(t_1,-\omega)\Phi(\omega)H_2^T(t_2,\omega)d\omega \tag{29}$$

where

$$H_1(t,\lambda) = \int_{-\infty}^{t} Z(t,s)F(s)e^{i\lambda s}ds \tag{30}$$

$$H_2(t,\lambda) = \int_{-\infty}^{t} Z(t,s)G(s)e^{i\lambda s}ds$$

Excitation satisfying Eq. (28) is treated in Ref. 3.

It is interesting to note that a direct relationship exists between the spectral densities of the input and output processes. First notice that for fixed τ, $Z(\tau,\zeta)$ is periodic in ζ with period $2T$. Hence assuming that $Z(\tau,\zeta)$ is absolutely integrable in τ, it has the same type of spectral representation as $R_{ff}(\tau,\zeta)$ namely

$$Z(\tau,\zeta) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_z(\omega,n)e^{i\omega\tau}e^{in\pi\zeta/T}d\omega \tag{31}$$

Inverting gives

$$\Phi_z(\omega,n) = \frac{1}{4\pi T}\int_{-\infty}^{\infty}\int_{-T}^{T} Z(\zeta,\tau)e^{-i\omega\tau}e^{-in\pi\zeta/T}\,d\zeta\,d\tau \tag{32}$$

Substituting Eq. (31) into Eq. (7), changing variables t_1, t_2 to τ, ζ, and simplifying using the appropriate Fourier orthogonality relations yields

$$R_{xx}(\tau,\zeta) = 4\pi^2 \sum_{j,k,l=-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_z[\lambda + (l+j)(\pi/T),j]\Phi_{ff}(\lambda,l) \tag{33}$$
$$\Phi_z^T[-\lambda + (l+k)\pi/T,k]\,e^{i\tau[(j-k)\pi/T+\lambda]}e^{i\zeta(j+k+l)\pi/T}d\lambda$$

Now substituting Eq. (33) into the expression for $\Phi_{xx}(\omega,n)$ [Eq. (24) with f replaced by x] yields after simplification

$$\Phi_{xx}(\omega,n) = 4\pi^2 \sum_{j,k=-\infty}^{\infty} \Phi_z\left[\omega + (n-j)\frac{\pi}{T},j\right]$$
$$\Phi_{ff}\left[\omega + (k-j)\frac{\pi}{T},n-k-j\right]\Phi_z^T\left[-\omega + (n-k)\frac{\pi}{T},k\right] \tag{34}$$

Conclusions

When viewed within the framework of weakly periodic nonstationary processes, the steady-state analysis of the random response of periodically time varying systems is quite similar to the well known analysis of time invariant systems subject to weakly stationary excitation. That is, all of the results derived herein reduce to well known results for linear time invariant systems subject to weakly stationary excitation. These results should prove useful in the further development of rotor vibration models.

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Simplified Conservation Laws for Finite-Difference Computations

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1. Introduction

FINITE-DIFFERENCE methods have proven to be powerful techniques for the solution of many fluid dynamical problems, particularly inviscid flows containing embedded shock waves. If the difference method is cast in conservation form, then the same difference equations can be applied at all grid points, including those at which shock waves are present. Although the interior structure given to the shock is artificial, the conservation form of the equations guarantees that the correct shock jump conditions are satisfied.

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In many applications, the full form of the inviscid conservation laws has been used even though the shock waves that are present are weak.[1-3] The full equations are used since they are known to provide the correct shock smearing behavior. However, the irrotational and isentropic assumptions may be used to simplify the equations in such a way that the correct shock smearing behavior is retained. For example, Cole[4] has recently noted that the steady nonlinear transonic potential equation can be cast in several conservation forms, and that one of these forms guarantees that the correct jump conditions are applied across shock waves. In the following Note, the full time dependent equations for potential flow are cast in a conservation form, which is demonstrated to contain the correct shock jump conditions to third order in shock strength.

2. Conservation System for Isentropic Flow

Consider a flowfield originating in an unsteady but isentropic freestream. The disturbances to the stream may be in the form of a weak shock or any isentropic gust. If all the shock waves that are embedded in the flowfield are weak, then the entire flow will be isentropic.

An isentropic state relation will hold between pressure and density at each point of the flowfield

$$p = p_0(\rho/\rho_0)^\gamma \tag{1}$$

for a polytropic gas with exponent γ. This condition guarantees that vortex lines are convected with the inviscid fluid particles. Consequently, the irrotationality condition

$$\operatorname{curl} \mathbf{u} = 0 \tag{2}$$

is commonly introduced in place of the momentum equation in most analytical work. This condition is difficult to use in time dependent, numerical studies, since it destroys the symmetry of the solution technique for the flow variables. However irrotationality can be used to simplify the momentum equation to a gradient of the Bernoulli equation. The system of flow equations is then simplified to

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 \tag{3}$$

$$\mathbf{u}_t + \nabla H = 0 \tag{4}$$

where H is the total enthalpy

$$H = [\gamma/(\gamma-1)]\, p/\rho + \tfrac{1}{2}q^2$$

This system has the formal appearance of a set of conservation laws, but aside from the continuity equation, the equations are not directly derived from any conservation principle. However, as is shown in the following paragraph, this set of equations does imply a valid isentropic form of the shock relations.

3. Weak Shock Jump Conditions

Let the flow be defined in a region Ω, and let the flow be smooth in Ω except across a surface Σ where the flow variables are discontinuous. If V is a subregion of smooth flow which is fixed in space then Eqs. (3) and (4) are derivable from integral laws of the form

$$\frac{d}{dt}\int_V \rho\, dV + \int_{\partial V} \rho\mathbf{u}\cdot\mathbf{n}\, dA = 0 \tag{5}$$

$$\frac{d}{dt}\int_V \mathbf{n}\, dV + \int_{\partial V} H\mathbf{n}\, dA = 0 \tag{6}$$

where $\mathbf{n}$ is the outward normal to V. The first expression is conservation of mass, of course, and, like all conservation principles, is valid even for regions V that contain Σ. However the second equation strictly holds only for regions where the flow is smooth. To find the shock jump conditions implied by Eq. (6), it is assumed that Eq. (6) behaves as a conservation law and holds for all regions V including those containing Σ. If the resulting jump condition is nonsensical, then Eq. (6) must be rejected as a suitable conservation law for numerical shock smearing applications.

Now let V be a pill box of small height ε normal to Σ, and let Σ' denote the portion of Σ contained in V. The integrals over V can be expressed as the sum of the integrals over the portions of V in front of and behind Σ. The time rate of change of an integral over V is then the rate at which material is swept from the front to the back by the motion of Σ plus the rate of change in each of the portions of V at the front and back of Σ. In each of these latter regions the flow variables are smooth, and in the limit of small ε the integrals vanish. Then if Σ is given by the equation $F(x, y, z, t) = 0$, its normal velocity will be given by $U_n = -F_t/|\nabla F|$, and the time rate of changes are

$$\lim_{\varepsilon\to 0}\frac{d}{dt}\int_V \rho dV = -\int_{\Sigma'} [\rho] U_n dA$$

$$\lim_{\varepsilon\to 0}\frac{d}{dt}\int_V \mathbf{u} dV = -\int_{\Sigma'} [\mathbf{u}] U_n dA$$

where the bracket notation denotes the jump across the discontinuity. The jump is taken in the sense of the normal to Σ (if the normal points out of region 1 and into region 2, then $[w] = w_2 - w_1$). The integral over the surface of V in Eqs. (5) and (6) collapses to an integral over the surface Σ', and in the limit of small ε Eqs. (5) and (6) require that

$$\int_{\Sigma'} (-[\rho] U_n + [\rho q_n])\, dA = 0$$

$$\int_{\Sigma'} (-[\mathbf{u}] U_n + [H]\mathbf{n})\, dA = 0$$

These surface integrals imply, for a well behaved Σ, the respective jump relations

$$[\rho\,(q_n - U_n)] = 0 \tag{7}$$

$$[-q_n U_n + H] = 0 \tag{8}$$

$$[-q_t] U_n = 0 \tag{9}$$

the last two expressions being normal and tangential components to Σ of the last vector relation.

These equations do imply a reasonable set of shock relations for weak shocks. The first is the usual continuity relation. The second equation is a reduction from the usual unsteady energy equation, which implies conservation of stagnation enthalpy. This interpretation is most easily seen by adding and subtracting the term $\frac{1}{2}U_n^2$ and expressing H as $h + \frac{1}{2}q^2$. After $\frac{1}{2}q_t^2$ is dropped by cancellation, there results for Eq. (9),

$$[h + \tfrac{1}{2}(q_n - U_n)^2] = 0$$

The third condition implies continuity of tangential velocity components for a moving shock and, in the limit, a stationary shock. However, in contrast to the complete jump conditions, tangential discontinuities are not permitted within the flowfield. For many problems, when the contact surfaces are weak, there is no difficulty in the use of these equations. However, tangential discontinuities cannot always be ignored in potential flows. For example, in the computation of the flow about an unsteady, three-dimensional lifting wing, the region would have to be cut at the trailing vortex system and special boundary conditions applied there. If these equations are supplemented by the isentropic state relation (1), a closed system of shock relations is achieved.

4. Transonic Shock Polar

When the transformation to coordinates attached to the shock is made, the shock relations of the previous section can be shown to reduce to the transonic shock polar. Consider an element of the shock line (dx_s, dy_s), the shock relations (7–9) imply the respective steady flow conditions

$$[\rho u]\, dy_s - [\rho v]\, dx_s = 0 \tag{10}$$

$$[H] = 0 \tag{11}$$

$$[u]\, dx_s + [v]\, dy_s = 0 \tag{12}$$

It is now assumed that the upstream and downstream states of the flow differ only by small quantities which will be denoted by tildas

$$\rho_2 = \rho_1(1 + \tilde{\rho}), \quad u_2 = u_1(1 + \tilde{u}), \quad v_2 = u_1\tilde{v}$$

where it has been assumed that the initial flow direction lies along the x axis. Elimination of the shock direction between Eqs. (10) and (12) gives

$$(\tilde{v} + \tilde{\rho}\tilde{v})\tilde{v} + (\tilde{\rho} + \tilde{u} + \tilde{\rho}\tilde{u})\tilde{u} = 0 \tag{13}$$

Now expand the energy Eq. (11) to second order using the isentropic relation to eliminate the pressure. Then taking $M_1 = 1$ in the second-order terms gives

$$\tilde{\rho} + M_1^2\tilde{u} + [\gamma - 1)/2]u^2 + \tfrac{1}{2}\tilde{v}^2 = 0 \tag{14}$$

By means of this form of the energy equation, the terms of the second bracket of Eq. (13) are seen to be of order $(1 - M_1^2)\tilde{u}^2$ and $\tilde{u}^3$. Consequently in transonic flow $\tilde{v}^2$ is negligible compared to $\tilde{u}^2$ and can be dropped from Eq. (14). Similarly $\tilde{\rho}\tilde{v}^2$ is negligible in Eq. (13), which after $\tilde{\rho}$ is eliminated reads

$$\tilde{v}^2 + (1 - M_1^2)\tilde{u}^2 - [(\gamma + 1)/2]\tilde{u}^3 = 0 \tag{15}$$

This equation is the usual transonic expansion of the exact shock polar

$$\tilde{v}^2 = \tilde{u}^2 \frac{(M_1^2 - 1) + [(\gamma + 1)/2]M_1^2\tilde{u}}{1 - [(\gamma + 1)/2]M_1^2\tilde{u}}$$

5. Conclusion

The equations of unsteady potential flow can be cast into conservation form for use in finite-difference applications. The conservation form provides the correct shock jump for shock smearing applications, and is accurate to third order in the shock strength.The most restrictive condition arises from the condition of tangential velocity continuity. In contrast to the results from the complete system of conservation laws, the simplified equations do not admit tangential discontinuities. This behavior must be expected from the imposition of the irrotationality condition throughout the finite difference mesh. Problems with strong tangential discontinuities must be treated by cutting the mesh system along the discontinuity and including it explicitly through boundary conditions. This restriction is not especially severe since important tangential discontinuities should be treated explicitly even with the full equations.[5] The artificial viscosity is usually so large that it diffuses the discontinuity very strongly within a few mesh intervals.

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The Effect of Angle of Attack on Boundary-Layer Transition on Cones

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Nomenclature

k = parameter related to circumferential gradient of circumferential velocity on the windward ray of a cone

$$k = \left(\frac{2}{3 \sin \theta_c}\right)\left(\frac{1}{V_e}\frac{\partial w}{\partial \Phi}\right)_{\Phi} = 0°$$

M = Mach number
Re_θ = Reynolds number based on momentum thickness
s_t = length to transition along a cone generator
V = velocity along a streamline
w = circumferential component of velocity
α = angle of attack
θ_c = cone half-angle
μ = viscosity
ρ = density
Φ = angular coordinate around the cone ($\Phi = 0°$; windward ray)

Subscripts

e = boundary-layer edge condition
∞ = freestream condition

PREVIOUS transition investigations on cones at angle of attack[1–3] have been mainly devoted to the windward and leeward rays and to angles of attack less than the cone half-angle. Although the results of these studies show a fairly consistent qualitative behavior (i.e., transition moves forward on the leeward ray and aft on the windward), a quantitative, parametric description has not been found. In an effort to satisfy the need for more data and to provide a definition of the parameters affecting transition the present study was initiated. The objectives of the investigation were to 1) obtain a detailed map of transition around a cone of angle of attack and 2) attempt a correlation of the results with existing data.

Tests were conducted in the Ames Research Center's 3.5 ft hypersonic wind tunnel at a freestream Mach number of 7.4. The models were 5° and 15° half-angle cones at angles of attack from 0° to 20°. The wall and total temperatures were nominally 305°K and 833°K, respectively, and total pressures ranged from 2.165×10^6 to 1.209×10^7 N/m^2. Boundary-layer transition was determined from heating rate distributions (deduced from thermocouple histories) as described in Ref. 4.

Transition Reynolds numbers were defined using boundary-layer edge conditions calculated by the method of characteristics.[5] Edge conditions for angles of attack greater than the cone half-angle were obtained using the following approximations. 1) The 15° cone edge, conditions at $\alpha = 20°$ were obtained by extrapolation from $\alpha \leqq 15°$. 2) The windward ray conditions on the 5° cone for $\alpha > 5°$ were calculated using characteristics by replacing the leeside with an ellipse so that the leeward ray was aligned with the freestream velocity vector. 3) Conditions on the leeward ray of the 5° cone at $\alpha = 6°$ were obtained by extrapolation from $\alpha \leqq 5°$. In formulating the transition Reynolds number the velocity along the local streamline was used in conjunction with the distance measured from the apex along conical rays.

The effect of angle of attack on local transition Reynolds number is illustrated in Fig. 1. It is obvious from the figure that the influence of angle of attack on transition Reynolds number is a function of meridian angle, Φ. For example, on the windward ray of the 15° cone transition Reynolds numbers show an initial, slight increase with α and then a decrease; whereas, leeward ray Reynolds numbers decrease rapidly with α. On the leeward ray the effect of α on transition is similar for both models, however, the windward ray results are considerably different.

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